Distinguishing Indistinguishables

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The third A. F. Parker-Rhodes Memorial Lecture

I. Introduction

When Clive Kilmister invited me to give this lecture, I thought about it for several days before accepting. In addition to being uncertain of my ability to attend this year, I was also undecided at to what I might talk about that would honor Parker-Rhodes while at the same time being a topic to which I might make some small contribution (or at least, not to babble too incoherently).

The title of this talk is some evidence of my decision to proceed; this will almost certainly not be a distinguished lecture, but it should not be indistinguishable from other lectures either. I apologize in advance if it should turn out that printed copies of the talk are not available, but I will nonetheless endeavor to keep this conceptual and philosophical so that pen and paper are not needed.

Parker-Rhodes gave great importance to the concept of indistinguishables. He found the treatment in standard mathematics to be superficial and full of assumptions. This situation caused his own work on the topic to require discussions which were subtle and hard to follow. This situation carried over into the formalism as well. So I feel that my interpretation of his work and to the extent to which I will indulge myself here may be in error, but is at least an honest attempt. This leads to a serious warning however: listen carefully and skeptically to quotations from The Theory of Indistinguishables lest you follow me too well and down the wrong path.

I do not intend to explain the Parker-Rhodes theory; that is a subject appropriate for a year long graduate course in foundations of mathematics. Instead the approach will be to introduce the key ideas behind Parker-Rhodes indistinguishables and then to tell a lot of stuff about other kinds of indistinguishables. In this way perhaps one can learn to distinguish these various sorts of indistinguishables (pun intended) when they are encountered.

II. Parker-Rhodes Indistinguishables

From a foundational point-of-view, Parker-Rhodes' indistinguishables are interesting in another way. As will be discussed shortly, indistinguishables are usually defined in terms of the properties of classes of indistinguishables.

saw them as underlying the physical plane (as distinct from the human and organic planes). He did not think of these as being reducible one to the other, although he clearly thought of them as layered one above the other. He therefore defined two types of indistinguishables: primary and secondary. A primary indistinguishable is a fortiori strictly unobservable. A secondary indistinguishable is an 'inscrutable' meaning that perhaps they only hide their differences from the observer and could be distinguished if the observer only knew how. We will return to this distinction below.

Parker-Rhodes felt that there were two key ideas behind his indistinguishables: first, how 'identicals', 'indistinguishables', and 'unequals' might be defined, and second, the notion of a triparitous relation based on this definition. The definition of classes of indistinguishables was defined so that whether entities were the same, different, or indistinguishable depended on how they contributed to the cardinalities of the classes to which they belonged. This idea property allowed Parker-Rhodes to define classes of indistinguishables with observable properties, but in which individual indistinguishables were not observable. Part of this notion is contained in the 'triparitous' parity-relations. Thus comparing indistinguishables led not only to the results 'identical' and 'distinct', but to 'twins', 'non-identical', 'bipar', and 'indistinct'.

This same idea of the non-uniqueness of negation is found in other areas of mathematics and logic: many-valued logics, non-distributive lattices, intuitionism, etc. With Parker-Rhodes however, it formed the basis for a different kind of mathematics; he found that his notation could not be interpreted uniquely without knowledge of the context, but managed to keep his sense of humor:

"...my notation will not be 'context-free'... This introduces a serious complication into the theory -- for it is a feature of all normal mathematics that it is couched in a context-free notation... One can't hope that this will help to popularize the theory."

Parker-Rhodes was well aware of the burden that this context-sensitivity placed both on himself and on his readers:

"Difficulties of exposition make themselves felt from the first, and it is not difficult to understand why the idea of indistinguishables has been so long neglected. Quite apart from the philosophical problems...these difficulties include not only, as is inevitable, a revised and more complex axiom-schema to replace the familiar rules about the substitutability of equals, but also a whole preliminary section of analysis which can ordinarily be passed over in silence. This concerns the semantics of mathematical notations."

When it came to doing something with indistinguishables, Parker-Rhodes defined two operations: correlation and predication. Correlation consists of identifying classes of entities by matching their respective cardinalities with known physical concepts. Predication consists of predicating numerical values to the class based on the correlation; for Parker-Rhodes this involved a much more complicated process.

It seems that Parker-Rhodes was also aware of the literature in regard to indistinguishables and found it wanting:

"It is therefore at first sight surprising that there exists no branch of mathematics, in which a third parity-relation, besides equality and inequality, is admitted..."

"The concept of what I here call 'indistinguishability' is not unknown in logic, albeit much neglected. It is mentioned, for example, by F. P. Ramsey ... who criticizes Whitehead and Russell... for defining 'identity' in such a way as to make inditinguishables identical."

As we shall see, Parker-Rhodes ideas are distinguishable from other definitions of indistinguishability.

III. CLASSICAL INDISTINGUISHABILITY

The notion of indistisnguishability is precluded from having a fundamental role in set-theoretic mathematics. The definition of a set forces each member to be distinct. This at once places the discussion of indistinguishables on a level different from that proposed by Parker-Rhodes.

There are a variety of ways in which one normally encounters indistinguishability in classical mathematics and logic. Perhaps the most common idea is that of an equivalence class; a class of entities are said to be equivalent under a particular relationship and thus form a class. This notion is easy to work with and is found in set theoretic discussions of indistinguishability. In this case, the indistinguishability is not fundamental; it is an abstraction based on ignoring the properties which make the members of the class distinct.

A variation on this kind of indistinguishability is found in statistics. Boltzmann statistics aries because the 'boxes' are equivalent under the distribution function. Variations occur in Bose-Einstein and Fermi-Dirac statistics where it is the equivalence of certain properties of the entities that are questioned rather than an equivalence under the counting operation.

IV. MULTIISETS

A number of attempts have been made to remove the restriction of member distinctiveness from set theory; these attempts are usually called multiset theories. Among the efforts along this

Biz ard follow the usual practice of defining the properties of the collection rather than the properties of the atoms of the collection. Manna defines the ways in which 'bags' can be combined -- a kind of combinatorics for classes of indistinguihables. Bliz ard follows the practice of defining the inference rules for multisets: this leads to rules, for example, for substitution of indistinguishables in place of the rules of substitution for equals.

While these approaches are highly satisfying from the formal point-of-view, I suspect they would leave Parker-Rhodes with the feeling that the cart was before the horse; if one thinks of indistinguishables as fundamental, it is disconcerting to be able to define them only after 'distinct' elements are defined.

V. RELATION THEORY

An approach which was pointed out to me by Pat Suppes involves defining a relation between atoms which is slightly different from an equivalence relation. In particular, this relation meets the usual requirements for an equivalence relation (symmetry, transitivity, and reflexivity) except for transitivity.

In some sense this is more satisfying that the classical and multiset ideas; it does deal directly with the atoms. But it like the other approaches, it leaves one without any understanding of how one arrives at the decision that two atoms are distinct or equivalent under some relationship.

VI. ORDERING OPERATOR CALCULUS

Without engaging in too much discussion, I would like to set forth some of the ideas underlying one more concept of indistinguishability. This is one I know more about since I have made it up as I went along. It's my own idea.

Like Parker-Rhodes, I think of indistinguishability as fundamental and consider the possible relationships between indistinguishables to be key. Unlike him, I place a stronger importance on process. Parker-Rhodes referred to the kind of indistinguishables found in the ordering operator calculus as inscutables and did not deal with them. In the ordering operator calculus, the 'reason' for an entity being an inscrutable is well-defind. It is based on the concept of computability: if there exists a decision procedure which can be completely represented within the system under consideration and which serves to distinguish two entities, then those entities are distinguishable. Otherwise, they are indistinguishable. For two entities to be identical, every property must be held in common and a deciion procedure must exist to identify the equivalence of each such property.

Two ideas are at work here. Like Parker-Rhodes theory, the ordering operator calculus is context-sensitive. Unlike Parker-Rhodes, I have insisted that syntax and semantics are essentially the same. In particualar, properties arise from the context in which an entity is embedded -- from the structure of the relationships which an entity has with other entities. This means that entities do not generally occur in isolation unless they are without properties. It follows immediately that all entities in isolation are indistinguishable; this is reminiscent of Parker-Rhodes Inchoative Plane.

However, the ordering operator calculus takes the idea a step further; if there is no decision procedure which would distinguish two entities then these can be treated as though they are in isolation. On the one hand, this means that we can not determine whether or not we are 'at the bottom of the heap.' One the other hand, it means that it doesn't matter; any 'bottom' will do.

Ordering operator calculus indistinguishability is context sensitive with a vengeance. It is not a context-sensitivity of notation but is manifest; it can not be removed except by considering an extremely restricted context.

These ideas remove the distinction between inscrutables and indistinguihables, between correlation and predication, between syntax and semantics. They place the definition of indistinguishability at the level of the atom rather than just the collection by refusing to entertain a reductionist doctrine in any form. Parker-Rhodes and I never had a chance to engage in detailed discussion about these differences in our theories, but we did agree on the intent. I can only hope that he would have approved of the 'correlations' and 'predications' which have been achieved in the interim. His influence on the work has been strong, even if subtle. One thing is certain: distinguishing indistinguishables takes great care -- and an effective procedure.